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BIOT'S VARIATIONAL PRINCIPLE  
IN  
HEAT CONDUCTION

by

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DEPARTMENT  
of  
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and  
APPLIED MECHANICS

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**12. Abstract:** The literature on Biot's variational principle for heat conduction and the thermodynamic foundations of the principle are reviewed. An additional example to those presented in a previous paper is given. This example treats the heating of slabs exposed to time-dependent heat fluxes and a specific example of a triangular heat pulse is presented in detail.

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## ABSTRACT

The literature on Biot's variational principle for heat conduction and the thermodynamic foundations of the principle are reviewed. An additional example to those presented in a previous paper is given. This example treats the heating of slabs exposed to time-dependent heat fluxes and a specific example of a triangular heat pulse is presented in detail.

## SECTION I. LIST OF SYMBOLS

$a_{ij}$	constant in entropy expression for neighboring thermodynamic state, Eq. (1)
$b$	slab thickness
$b_{ij}$	thermodynamic coefficients in relations between forces and fluxes, Eq. (4)
$c$	specific heat
$dv$	volume element
$D$	dissipation function
$D^*$	operational form of dissipation function
$\text{div}$	$\nabla \cdot$
$\exp(x)$	exponential function of $x$
$F$	defined by Eq. (26)
$g$	defined by Eq. (25)
$h$	heat transfer coefficient; heat flow from System I to System II
$n_2$	heat flow from energy reservoir to System I
$\vec{H}$	heat flux vector field
$H_n$	value of heat flux field at the surface
$k$	coefficient of thermal conductivity
$p$	pressure
$q$	heat flux
$q_1$	surface temperature
$q_2$	penetration depth
$q_3$	surface temperature



$q_i$	generalized coordinate
$Q_2$	thermal force
$Q_i$	generalized thermal force
$S$	entropy: entropy of System I
$S'$	entropy of total system
$S_c$	defined by Eqs. (13)
$S_r$	defined by Eqs. (13)
$S_2$	entropy inflow to System I
$S_{II}$	entropy of constant temperature reservoir
$S_{III}$	entropy of energy reservoir
$t$	time
$t_p$	penetration time
$T$	temperature
$T_r$	reservoir temperature
$U$	internal energy
$U_c$	defined by Eqs. (13)
$U_i$	defined by Eqs. (13)
$V$	thermal potential function
$V_c$	defined by Eqs. (13)
$V_i$	defined by Eqs. (13)
$x$	spatial coordinate
$X_i$	generalized thermodynamic force
$z$	defined by Eq. (26)
$\alpha$	thermal diffusivity, $k/c$
$\delta$	variation

$\eta$	dimensionless penetration depth
$\theta$	temperature difference measured from equilibrium temperature
$\tau$	dimensionless time
$\tau_p$	dimensionless penetration time
$\theta_1, \theta_2$	dimensionless surface temperature
$\psi$	surface temperature parameter
$(\cdot)$	$d/dt$
$\nabla$	$\underline{i}(\partial/\partial x) + \underline{j}(\partial/\partial y) + \underline{k}(\partial/\partial z)$
$(\underline{\quad})$	vector quantity

## SECTION II. INTRODUCTION

The high speeds of modern aircraft and guided missiles have introduced new problems into the design of structural elements for such vehicles. These problems are a consequence of the large heat input to the structure arising due to the increase in temperature of the air in immediate contact with the surface of the vehicle. This increase in temperature is caused either by shock compression at the blunt leading edge or by friction in the boundary layer. Similar type of heat inputs also arise due to the flow of high energy gases in rocket motors and nuclear reactors although the source of heat is physically different. In all of these situations the temperature rise in the structural components must be accounted for in the design of the entire structure. This requires among other problems the determination from a heat conduction analysis of the temperature distribution in the structure under the known thermal inputs.

However, the large heat inputs and the accompanying high temperatures will necessitate a consideration of the temperature dependence of the thermal properties of the heated body. The introduction of these two effects will make the governing heat conduction equation and boundary conditions nonlinear, a problem which can not be treated by the usual classical methods of heat conduction analysis. Refs. [1, 12]. In such

situations, approximate analytical or numerical methods are required in order to obtain a solution. A discussion of some of the analytical and numerical methods of solution was presented in Refs. [12, 13, 14].

One of the most important of the analytical methods of solution for heat conduction is the variational principle introduced by Biot, Refs. [2 to 4]. Biot has approached the problem from a fundamental standpoint as a variational principle in thermoelasticity which treats the coupled elastic and thermal fields directly. If the thermal effects are ignored, the classical variational theorems of elasticity are obtained from the variational formulation. If the elastic effects are ignored, a variational form of the heat conduction equation is obtained.

This principle was reviewed from a mathematical viewpoint for the special case of heat conduction in Ref. [13]. In this reference, the principle was discussed for the case in which the boundary condition involved the heat flux. A method for including such boundary conditions which had not been treated before by the variational principle was introduced and the variational principle was then applied to a number of different one-dimensional heat conduction problems. The problems treated included non-linear heat flux boundary conditions, time-dependent aerodynamic heating, and the heating of bodies with temperature-dependent material properties. The results obtained for these

examples demonstrated the applicability of the variational principle to problems with specified heat flux boundary conditions.

In the present paper, which complements Ref. [13], the thermodynamic foundations of Biot's variational principle are reviewed together with previous work on variational methods for heat conduction. Since Biot's variational principle includes the field of thermo-elasticity, a discussion of other variational principles obtained from Biot's formulation is also presented. In this way, the principle as applied to the analysis of heat conduction problems is put in the proper perspective as a special case of a more general principle.

Section III contains a review of the literature on previous work employing variational principles for heat conduction and on Biot's variational principle. Section IV reviews the thermodynamic foundations of the general variational principle and its form for the special case of heat conduction.

The method for including heat flux boundary conditions that was introduced in Ref. [13] is applied in Section V to the problem of the heating of a finite slab with an arbitrary time-dependent heat input. The heating of a slab by a triangular heat pulse is then discussed in detail.

### SECTION III. LITERATURE REVIEW

A discussion of variational principles in thermo-elasticity and heat conduction is presented in this section. Although the present paper is concerned mainly with heat conduction, the discussion of the variational principles in thermo-elasticity presented below provides a basis for the discussion of Biot's variational principle, Refs. [2 to 11]. This principle is applicable to the general field of thermo-elasticity and hence includes the heat conduction analysis as part of the complete formulation. Therefore, by presenting the discussion in this way, the variational principle for heat conduction alone is viewed in its proper perspective as a specific case of a more general principle. In addition to these discussions reference will be made to other approximate methods of solution for the heat conduction equation.

#### Variational Principles in Thermo-Elasticity

Biot, Refs. [4, 7], has introduced a variational principle which has application to the field of thermo-elasticity and in fact to more general thermodynamic systems. The field of thermo-elasticity includes, in the most general sense, the effects of coupling between the temperature and elastic fields. It is this variational principle, restricted to the specific field of heat conduction, that will be discussed in

this paper.

The general variational principle was derived by Biot on the basis of thermodynamical arguments applied to a physical system. The state of this system was defined by generalized state variables and Onsager's reciprocal relations were applied to obtain Lagrangian equations for the time histories of the generalized coordinates. Refs. [2, 3, 7]. This method of deducing the equations will be discussed in Section IV. Once the general equations were established, a variational principle equivalent to these equations was introduced. This variational principle was applied to the general case of thermo-elasticity mentioned above, Refs. [4, 7], and to the case where the elastic effects were not considered, Refs. [5, 6, 9]. The latter case, of course, applies to the field of heat conduction. An additional field of application of the general variational principle which has been treated by Biot is viscoelasticity, Refs. [10, 11]. In what follows in this section, however, the principle will be discussed in relation to thermo-elasticity which includes, in the general sense, the field of heat conduction.

Biot has shown that the governing equations for equilibrium of a thermo-elastic solid can be expressed in terms of a variational principle under specified constraint conditions, Ref. [4]. In particular, if the equation of state together with conservation of energy is imposed on the variational function,

the equilibrium equations and Fourier's law of conduction are obtained as the Euler differential equations. The latter equation with energy conservation yields the coupled heat conduction equation. This principle is analogous to the principle of Minimum Potential Energy in Elasticity, Refs. [15, 16], in which the stress-strain relations are imposed and the equilibrium equations are the corresponding Euler equations.

A Complementary Energy Principle was formulated by Herrmann, Ref. [17], and leads to the stress-displacement equations together with the conservation of energy expression under the assumption that the varied stress and temperature fields satisfy the equilibrium equations and Fourier's law of conduction. The emphasis on Fourier's law of conduction is not clearly stated by Herrmann; this point, however, is particularly important since it forms the basis of the application of the variational principle in heat conduction. This was noted in Ref. [13]. It is seen from the above that this principle is analogous to the principle of Minimum Complementary Energy, Refs. [15, 16], in which the equilibrium equations restrict the arbitrary stress variations to yield the compatibility equations.

Herrmann, in two later papers, Refs. [18, 19], has considered an extension of Reissner's variational principle, Refs. [20, 21, 22], into thermo-elasticity in which all the field equations are obtained as a consequence of a variational principle. The principle has been outlined for the case of a uniaxial state



of stress, Ref. [18], and generalized to a three-dimensional anisotropic body, Ref. [19].

However, this extension to a more general variational principle introduced an additional variable called the "thermal dis-equilibrium force" which has not been employed by Biot or by Herrmann in setting up the separate variational principles for thermo-elasticity mentioned above. This force has the form used in irreversible thermodynamics since it is related to the gradient of the temperature field. Moreover, it provides the connecting link between the law of conduction and the energy equation which leads to the coupled heat conduction equation.

With the introduction of this independent variable, the governing field equations were obtained from the functional expression given by Herrmann in a manner similar to that outlined by Reissner. Comparison with neighboring states which do not satisfy the variational principle showed that the principle is that of a stationary value problem rather than a true maximum or minimum; again this paralleled the development presented by Reissner.

An additional formulation of a variational principle to include thermo-elasticity, plasticity and creep deformation has been presented by Besseling, Refs. [23, 24]. This variational principle was stated in terms of the physical displacement and entropy displacement fields and leads in the general

case to a stationary value principle. The entropy displacement was the same as that used by Dier, Ref. [4]. Furthermore, an application of this variational principle to a problem of structural damping was presented.

#### Variational Principles in Heat Conduction

Weiner, Ref. [25], presented a method of solution for the heat conduction equation which combined the standard Laplace technique with that of the Galerkin technique (see, for example, Refs. [15] and [16]). In this formulation, the transformed heat conduction equation in the Laplace variable was solved approximately by the Galerkin technique. The approximate solution obtained in this manner was then transformed back to recover the time variable. The conclusion drawn by Weiner in applying this method to some simple examples and to the flow of heat in an angle section was that for problems of physical interest considerable computational labor is required.

Washizu, Ref. [26] has presented a variational principle which introduced a temperature field that was the mirror image of the physical temperature distribution. This artificial temperature can be interpreted as that corresponding to a negative thermal conductivity. The method was applied by assuming a polynomial temperature distribution to be used in the variational function from which the coefficients of the polynomial were found. An example involving the heating of a finite slab with fixed sur-

face temperatures showed poor agreement with the exact solution for an appreciable time range, in particular, for short times.

Rosen, Ref. [27], formulated a variational principle based upon a modification of Onsager's principle of minimum dissipation. The functional expression introduced was expressed in terms of the temperature gradients and the time derivative of the temperature. This functional expression was varied with respect to the temperature with the time derivative held fixed. Appropriate boundary conditions were enforced so that the heat conduction equation was obtained from the variational function. While the variational principle is consistent within the framework presented, a logical inconsistency does exist if consideration is given to conservation of energy. No applications of this principle were presented by Rosen so that no statements as to applicability of the principle can be made.

A variational principle almost opposite to that presented by Rosen was given by Chambers, Ref. [28]. In this variational principle based upon work of Verivel, Ref. [29], a variational function was expressed in terms of the temperature, the time rate of change of the temperature, the temperature gradient and the heat flux. The variation was performed with respect to the time derivative of the temperature while the temperature itself remained unchanged. However, although the heat conduction equation and the appropriate boundary conditions were obtained in a formal manner, a physical inconsistency exists in this prin-

ciple. This is due to the introduction of the heat flux without considering its relation to the time rate of change of the temperature through the expression for energy conservation. No illustrations of the principle are presented.

The above two variational principles illustrate the inconsistencies that may occur when a variational principle is introduced without an underlying physical concept. In particular, the second principle is not correct from a physical point of view. On the other hand, the variational principle introduced by Biot is motivated by strong physical reasoning. This principle which was discussed above has, in addition, the advantage of generality. However, as was mentioned before, the variational principle will be applied in this paper to heat conduction processes only. This is equivalent to neglecting the elastic effects in the thermo-elastic variational principle.

The motivation for the principle from thermodynamic considerations will be presented in Section IV. An additional discussion of the thermodynamic principles in Biot's theory can be found in Ref. [30]

The variational principle was discussed from a mathematical point of view in Ref. [13]. That is, the mathematical steps showing the equivalence of the variational principle to the heat conduction equation was presented. Similar discussions of the variational principle were given in Refs. [31, 32].

In addition, the problem of the determination of the temperature distribution in a flange-web combination (angle section) which was treated by Biot, Ref. [5], was further discussed in Refs. [33,34]. An extension of the variational method for this problem to two dimensions was given by Levinson, Ref. [35]. These references, with the exception of Refs. [13,35], do not present any new formulations or results of the variational principle. However, the principle was modified for the case of both transient and steady forced convection heat transfer in Refs. [36 to 39].

Reference [37], Part I, contains a formulation of a variational principle for the transient heat convection equation analogous to that formulated by Biot for the heat conduction equation. Gupta, Ref. [38], has extended this principle for convection to the case where the medium is anisotropic. Both of these principles require a modification of Biot's work to account for the motion of the medium and for the viscous dissipation. If the latter effect is neglected, the resulting heat convection equation is identical to the heat conduction equation for a moving medium. In this case the dissipation function in the principle is modified by expressing the flux condition in terms of the velocity of the medium.

Agrawal, Ref. [37], Part II, introduced a variational principle for steady heat convection in channel flow under the assumption that the effect of axial conduction can be neglected. Again this variational principle is similar to that of Biot's. Agrawal, in a later paper, Ref. [36], extended his previous principle to account for the vanishing of the velocity at the walls of the channel.

The restriction of negligible axial heat conduction has been set aside in the work of Gupta, Ref. [39], and a variational principle has been developed for the complete steady heat convection equation for channel flows. In this work, as well as in the above references for convection, Lagrangian type of equations have been formulated for the thermal flow field using the concepts of thermal potential, dissipation function and generalized force. The use of these concepts clearly shows the relation of this work on convection to the work of Biot in heat conduction.

Citron, Refs. [40, 41], has discussed Biot's principle with reference to ablation problems. In Citron's work the variational principle was expressed in terms of a functional expression involving the heat flux and the temperature gradient. The variation was carried out with respect to the heat flux with the temperature gradient held fixed. This particular method, however,

is not consistent with conservation of energy. That is, any variations in the heat flux must be related to variations in the temperature gradients as a consequence of conservation of energy.

Citron, Ref. [40], also presented an application of Galerkin's method (see above) to the ablation problem. This method was also presented in Ref. [32], and it is an extension of the Galerkin technique to the case where the approximate solution is presented as any function of the time-dependant arbitrary parameters. An illustration of this method is given in Ref. [32].

#### Other Approximate Methods in Heat Conduction Analysis

The remaining portion of this section will discuss additional approximate methods that have been used in heat conduction analysis. Although numerical methods have been most useful for the determination of temperature distributions, they will not be discussed here. However, specific numerical methods were discussed in Ref. [13] together with applications of Biot's variational principle to prescribed physical problems. While these numerical methods may have certain advantages if computational assistance is available, they require in any subsequent thermal stress calculation, for example, the continued application of numerical methods. This, together with the temperature problem may become a formidable undertaking. In addition, if an analytical solution is possible,

it may not be in a convenient form for numerical evaluation.

Therefore, it may be desirable in these cases, for the purposes of practical computation, to obtain a simple approximate analytical expression for the temperature distribution.

Goodman has introduced an approximate method called the heat balance integral technique for the solution of the heat conduction equation; this method has been used in many studies of heat conduction problems, (see Ref. [13] for a list of references.) The essential idea of this technique is to satisfy the integrated heat conduction equation with an assumed spatial temperature profile that must satisfy the prescribed boundary conditions. Many of the results obtained in Ref. [13] using Biot's variational principle were compared with this heat balance technique.

Green, Ref. [42], has presented an expansion method for a general form of the heat conduction equation which is similar to the Galerkin method. An application of a modification of this method, Ref. [32], was mentioned previously with reference to the Galerkin technique.



#### SECTION IV. THERMODYNAMIC FOUNDATIONS OF THE VARIATIONAL PRINCIPLE

The thermodynamic foundations, Ref. [7], which lead to the general form of Biot's variational principle are reviewed in this section. After the general form of the principle is developed, the discussion will be restricted to the foundation of the heat conduction variational principle.

The major points in the development of the general variational principle are the introduction of Onsager's reciprocal relations, Refs. [43 to 45], and the determination of the expressions for and the meanings of the dissipation function and the thermal potential. Of particular importance is the determination of the entropy of the total system in terms of the entropy and the internal energy of the primary sub-system and the temperature of the heat reservoir. This leads to an expression for the generalization of the Helmholtz free energy for a system at a non-uniform temperature; this expression is called the "Generalized Free Energy" or "Thermal Potential" but the latter term will be used in the subsequent discussions. The equations for the state variables of the system are first developed for an isolated system and then for a system under external forces. The type of forces considered will be of a thermal nature since the final objective is to

apply the results to the analysis of heat conduction problems. However, the applicability of the method for the determination of forces in other situations is apparent and this has been discussed by Biot, Refs. [3,7]. Once the formulation for the system has been established in the sense that the Lagrangian equations for the time histories of the state variables have been found, a variational principle equivalent to these equations is immediately obtained. This variational principle restricted to the special case of thermal fields leads to the heat conduction equation for the temperature. In this way a variational principle motivated by thermodynamic arguments can be established in a consistent manner for heat conduction processes.

#### Thermodynamic System

The entire system to be considered consists of a primary subsystem, called System I, to which a heat reservoir, called System II, at a constant temperature  $T_0$ , is connected. The entire system consisting of both the primary system and the heat reservoir is assumed to be isolated.

Initially, the system is in an equilibrium state for which the values of the entropy and internal energy are taken as equal to zero. Deviations from the equilibrium state are expressed in terms of  $n$  generalized coordinates  $q_i$  which are also zero in the equilibrium state. These  $n$  generalized coordinates are state variables defining the state of the entire

system at any instant of time. The value of the equilibrium temperature is  $T_r$  and the local deviations from this equilibrium value will be denoted by  $\theta$ ; these deviations are considered as state variables. The system is in addition assumed to be linear in the sense of irreversible thermodynamics.

This requires that the fluxes in the system are linearly related to the corresponding forces, for example, that the heat flow is related to the temperature gradient. In addition, production of entropy in the system must be written as a product of the fluxes and the corresponding forces. A review of these basic macroscopic concepts for the thermodynamic theory of irreversible processes was given by Miller in Ref. [43]. Miller discussed the foundations of irreversible thermodynamics, which are also discussed in Refs. [43 to 45], and presented the basic assumptions and motivation of the theory.

The system at first will be free from any external applied forces. Since the entropy of the system at equilibrium has a maximum value which is conveniently set equal to zero, the entropy of any neighboring state will be less than zero. This value of entropy can be expressed in terms of the state variables  $q_i$ . That is, the entropy at any neighboring state with state variables  $q_i$  can be written in the form

$$\tau_r S' = (1/2) a_{ij} q_i q_j$$

$$a_{ij} = a_{ji} \quad (1)$$

where  $S'$  is the entropy of the entire system, the  $a_{ij}$  are the values of the appropriate derivatives of the entropy evaluated at the equilibrium state and the  $q_i$  are the state variables. The summation convention of summing over repeated indices is used in this equation and in the equations to follow.

As a consequence of the deviation from the equilibrium state, restoring forces are set up within the system. It is assumed that these forces are linearly related to the time rate of change of the state variables, the fluxes,  $q_i$ . In addition it is assumed that the linear relations satisfy the Onsager hypotheses, Refs [43 to 45]. This requires that the entropy production in the system which arises as the system is returning to the equilibrium state be written in the form

$$\tau_r (dS'/dt) = X_i q_i \quad (2)$$

or that

$$T_r(AS'/\partial q_1) = X_1 \quad (3)$$

This latter equation states that the forces are equal to the corresponding gradients of the entropy in the non-equilibrium state, the factor  $T_r$  is introduced for convenience. If the forces  $X_1$  are now expressed in terms of the fluxes  $q_1$  in the following form:

$$X_1 = T_r(AS'/\partial q_1) = b_{1j} q_j \quad (4)$$

Onsager's principle leads to the result

$$b_{1j} = b_{j1} \quad (5)$$

These relations interconnect the various thermodynamic processes taking place within a thermodynamic system and they are called the Onsager reciprocal relations, Refs. [43 to 45]. The limitations of these relations are also discussed in the above references. A review of the experimental evidence for these reciprocal relations in macroscopic linear relations is given in Ref. [47]. The experimental evidence given indicates that the Onsager reciprocal relations are satisfied in most applications.

In view of Eq. (5), a quadratic function  $D$  exists such that

$$X_i = b_{ij} q_j = \partial D / \partial q_i$$

where

$$D = (1/2) b_{ij} q_i q_j \quad (6)$$

The physical significance of the function  $D$  follows immediately from Eq. (2) which expresses the rate of entropy production as a function of the forces and fluxes  $q_i$ ,

$$T_r (dS'/dt) = X_i q_i = b_{ij} q_j q_i = 2D$$

Therefore, the function  $D$  is  $(1/2)T_r$  times the rate of entropy production in the system; this function is called the dissipation function. The quadratic form of  $D$  expressed by Eq. (6) is then positive definite since the rate of entropy production is always positive.

In addition to expressing the force  $X_i$  as a derivative of the dissipation function, the entropy of the system is set equal to a function  $V$  called the thermal potential such that

$$V = - T_r S' = (1/2) a_{ij} q_i q_j \quad (7)$$

Therefore, Eq. (3) together with Eq (6) can be written in the following form:

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = 0 \quad (8)$$

These equations are in Lagrangian form for the  $n$  state variables  $q_i$ . They are analogous to the Lagrangian equations for a mechanical system in which  $V$  is the potential energy,  $D$  is the dissipation function and the  $q_i$  are the generalized displacements. The external forces acting on the mechanical system in this case are equal to zero. However, the right hand side of Eq. (3) is not zero if external forces are applied. A similar result would be expected by analogy for the thermal system under external forces. However before starting on that case a discussion of the significance of the function  $V$  will be presented.

The significance of the thermal potential function  $V$  can be found by writing the entropy of the entire thermodynamic system in terms of the entropies of the two subsystems. That is, the entropy  $S'$  of the total system is

$$S' = S + S_{II} \quad (9)$$

where  $S$  is the entropy of the primary sub-system and  $S_{II}$  is the entropy of the constant temperature reservoir. During the process of reaching the equilibrium state, heat will flow in general between the primary system and the reservoir. If this heat flow from System I to System II is  $h$ , then conservation of energy for System I gives

$$U = -h \quad (10)$$

where  $U$  is the internal energy of System I. At equilibrium both  $h$  and  $U$  are equal to zero. If external forces are acting on System I, Eq (10) will be modified by additional work terms; this will be treated subsequently.

Since the heat flow into the constant temperature reservoir is known, the entropy  $S_{II}$  is found in terms of the internal energy of the primary system, that is,

$$S_{II} = \frac{h}{T_r} = \frac{-U}{T_r} \quad (11)$$

Therefore, from Eqs (7), (9) and (11), it follows that the thermal potential function  $V$  can be written in the form

$$V = -T_r S' = U - T_r S \quad (12)$$



Equation (12) is similar to the form of the Helmholtz free energy for the System I except for the presence of the reservoir temperature  $T_r$ . Since the temperature of the primary system is not specified, and may have any arbitrary distribution, this form has greater applicability than the usual free energy. Biot, Ref. [2,7], calls this function the generalized free energy. This form of the thermal potential will be used later in the discussion of external forces.

An additional form of the function  $V$  in terms of temperature can also be found and it has a more convenient form for physical applications than that given above. In this expression the thermal potential is considered as the sum of two parts: (1) a value obtained when all the state variables are varied except the temperature which is held fixed and (2) a value obtained when the temperature is varied and the remaining state variables are held fixed. This splitting of the function  $V$  into two parts can be written in the form

$$V = V_r + V_c \quad (13)$$

where

$$V_r = U_r - T_r S_r$$

$$V_c = U_c - T_r S_c$$

These latter equations express each component of  $V$  in terms of the corresponding values of the internal energies and entropies for isothermal changes,  $U_T$  and  $S_T$ , and for constant state variable changes,  $U_C$  and  $S_C$ . That is,  $V_T$  is the value of the thermal potential for an isothermal process at temperature  $T_T$  while all the other state variables are varied, an analogous statement applies for  $V_C$ .

The term  $V_C$  can be explicitly expressed in terms of the temperature  $T_T + \theta$  of System I. This value of  $V_C$  for the entire System I is

$$\begin{aligned} V_C &= \int [U_C - T_T S_C] dv \\ &= \int \left[ \int_0^\theta c d\theta - T_T \int_0^\theta \frac{c d\theta}{T_T + \theta} \right] dv \end{aligned} \quad (14)$$

where  $c$  is the value of the specific heat obtained when all state variables except the temperature are held fixed. The first part of this expression follows from the internal energy term while the second term equal to the entropy of the system is the integral of the heat input over the temperature. The integration is performed over the volume of System I. Equation (14) can also be written in the form

$$V_C = \int \left[ \int_0^\theta \frac{c d\theta}{T_T + \theta} \right] dv$$

If  $\theta \ll T_r$ , that is, small departures from the equilibrium state,  $V_c$  can be written as

$$V_c = \frac{1}{2} \int \frac{c\theta^2}{T_r} dv \quad (15)$$

Therefore, the function  $V$  can be written in the form

$$V = V_r + \frac{1}{2} \int \frac{c\theta^2}{T_r} dv$$

where  $V_r$  is the value of  $V$  for an isothermal process. For example, in thermo-elasticity the value of  $V_r$  will be the value of the isothermal free energy or strain energy integrated over the volume and  $c$  will be the value of the specific heat at constant strain. In this way, the effects of the temperature and the remaining state variables can be considered separately.

Now that the significance of the thermal potential has been discussed, the introduction of external disturbing forces can be considered in order to formulate the complete Lagrangian equations for the state variables in the most general case.

#### Thermodynamic System: External Forces Applied

The equations for the time histories of the state variables of the thermodynamic system were given for the case of zero external forces by Eqs. (8). In order to obtain the form of the equations for the case where external forces are acting on System 1, the entropy of the total system will again be expressed in terms of the entropy and the internal energy of System 1. This will parallel the discussion of the significance of the thermal potential function  $V$ . However, in

this case conservation of energy will introduce the effects of the external forces. Consider a heat reservoir at temperature  $T_r = \theta$  adjoining the primary System I and let the heat flow from this reservoir be equal to  $h_2$ . Conservation of energy for System I is then expressed by

$$U = h_2 + h \quad (16)$$

where  $h$  is again the heat flow from System I to the System II. The entropy  $S'$  of the total system is

$$S = S + S_{II} + S_{III}$$

where  $S_{III}$  is the entropy of the reservoir at the temperature  $T_r = \theta$ . This equation can be written in the form

$$S = S + (h_2/T_r + \theta) + (h/T_I)$$

which becomes upon introduction of Eq. (16)

$$S' = S + (h_2/T_r + \theta) + h_2[(1/T_r) - (1/T_r - \theta)]$$

if  $0 \ll \theta$  it then follows that the entropy of the total system

$$S' = S - (U/T_r) + (h_2\theta/T_r)$$

The last term in this expression will be denoted as  $S_2\theta$  where the term

$$S_2 = (h_2/T_r)$$

will be called the entropy inflow to System I. Therefore, the expression for the entropy  $S'$  of the total system can be written in the form

$$T_r S' = -V + S_2\theta \quad (17)$$

where

$$V = U - T_r S$$

The term  $S_2\theta$  is similar to a generalized work expression in the Lagrangian formulation of the governing equations for a mechanical system. The quantity  $S_2$  is the force function and  $\theta$  is the conjugate displacement. Similar expressions for the work quantities of a thermal system in terms of physical forces, concentration gradients, etc., can be determined and some of these additional cases are outlined in Ref. [7]. In view of this, it is possible to write Eq. (17) in the general form

$$T_r S' = -V + Q_1 q_1$$

where  $Q_1$  is the generalized force conjugate to the generalized displacement  $q_1$ . For example, the generalized force conjugate to the temperature, as outlined above, is the entropy flow.

Onsager's relations can be applied to the complete system consisting of the primary system, the heat reservoir at the equilibrium temperature  $T_r$  and the energy reservoirs equivalent to the applied external forces. This application is similar to the case where no applied forces are acting on the system. In view of Eqs (4) and (6), the equations for the state variables take the form

$$T_r \frac{\partial S'}{\partial q_1} = \frac{\partial D}{\partial q_1}$$

or

$$\frac{\partial V}{\partial q_1} + \frac{\partial D}{\partial q_1} = Q_1 \quad (18)$$

Equations (18) are the governing equations for the state variables  $q_1$  of the thermal system. These equations are, to repeat the statements made above, similar to the equations of a mechanical system in which  $V$  is the potential energy,  $D$  is the dissipation function and the  $Q_1$  are now the external forces acting on the system. An additional way of writing Eqs (18) is

$$\frac{\partial D}{\partial q_1} = 2 \quad \frac{\partial V}{\partial q_1} = \frac{\partial}{\partial q_1} \left( \frac{T_r}{2} \frac{dS'}{dt} \right)$$

which expresses the forces, both external and internal, in terms of the entropy production in the system. Additional properties

of Eqs. (18) are also discussed in Refs. [2, 3, 7] but these will not be discussed here. Instead, the equivalence of the equations to a variational principle will be shown with a view toward obtaining a variational principle for heat conduction analysis.

The function  $D$  can be written in the following operational form

$$D^* = (1/2) p b_{ij} q_i q_j$$

where

$$p = d/dt$$

If the dissipation function is written in this form, Eqs. (18) can be written in the variational form

$$\delta V - \delta D^* = Q_i \delta q_i \quad (19)$$

The variation is with respect to the generalized coordinates  $q_i$  and the operator  $p$  is treated as a constant when calculating the variation. This form of the variational principle is convenient in some cases, but the variational principle can be established also on the basis of the definitions of the dissipation function and thermal potential, Ref. [4]. This will be the method for the determination of the form of the variational principle in heat conduction analysis to be discussed next.

### Heat Flow

The Lagrangian equations obtained above are equivalent to the variational principle given by Eq. (19). Instead of continuing the study of a thermodynamic system in complete generality, the particular case for heat flow alone will be studied. That is, the main concern will be with the thermal field  $\theta$ . In particular, the expressions for the dissipation function and thermal potential will be found.

The thermal potential for this restricted case follows immediately from Eq. (15), and is

$$V = \frac{1}{2} \int \frac{c\theta^2}{T} dv$$

The dissipation function  $D$  is expressed in the form

$$D = 1/2T^2 \dot{\theta}^2 \quad (\text{Rate of Entropy Production})$$

Therefore it is necessary to express the rate of entropy production in terms of known quantities of the thermal system.

This can be done by applying the first and second laws of thermodynamics to such a system. The first law is

$$\frac{dU}{dt} = \dot{Q} - \dot{W}$$



where  $U$  is the internal energy of the system and  $\underline{H}$  is the heat flux vector field, that is, the heat flow per unit time. The second law gives

$$T \frac{dS}{dt} = \frac{dU}{dt} = - \operatorname{div} \underline{H} \quad (20)$$

where  $T$  is the temperature and  $S$  is the entropy. Equation (20) can be written in the form

$$\frac{dS}{dt} + \operatorname{div} (\underline{H}/T) = - (\underline{H}/T^2) \cdot \nabla T \quad (21)$$

The left hand side of this equation is interpreted as the change in entropy of a unit volume plus the entropy flow across the surface of this volume. The net change in entropy for the unit volume is then given by the right hand side of Eq. (21); this term is the entropy production in the volume. Furthermore if the thermal force,  $\nabla T$ , is linearly related to the flux field such that

$$\underline{H} \sim - \nabla T$$

the entropy production term is proportional to the square of the heat flux field. The minus sign in the above expression arises since the heat flow must be opposite to the temperature

gradient. Therefore, the dissipation function can be expressed in terms of the square of the heat flux field

The remaining term discussed with reference to the general thermodynamic system was the generalized force  $Q_1$ . This force, in the case of a purely thermal field, is immediately known from the previous discussions and it is equal to the heat flow field divided by  $T_r$ . The factor  $T_r$  used above can be dropped since it is common to all terms in Eqs. (18).

The functions in the Lagrangian equations can now be expressed in terms of the temperature field and the time rate of change of the heat flow field. Once so expressed, as was done in the above paragraphs, the discussion for the general thermodynamic system indicates the form of a variational principle which is applicable to a thermal system, in particular, to the study of heat conduction in the system. This variational principle was given by Biot, Refs. [5, 6, 9] and it was shown to be equivalent to the heat conduction equation. Once this equivalence was established, the heat flow field related to the temperature field by conservation of energy was expressed in terms of generalized coordinates. The variational principle was then shown to lead to Lagrangian equations for the generalized coordinates of the temperature field. In this way the motivation behind the heat conduction variational principle as being related

to the Lagrangian representation of a general thermodynamic system is shown.

Further discussions of the principle for heat conduction were given in Ref. [13] together with a number of applications. A further application of the principle will be given in the next section.

## SECTION V. THE HEATING OF SLABS EXPOSED TO TIME-DEPENDENT HEAT FLUXES

The problem of the determination of the temperature distribution in one-dimensional slabs heated by arbitrary time-dependent heat fluxes has received much attention in the last few years. This interest has been engendered by problems arising in the design of structural components for re-entry vehicles. In these design situations, the surface temperature is small compared to the stagnation temperature of the air and thus can be neglected in the usual aerodynamic heating rate. This assumption then reduces the heating rate at the surface to an arbitrary function of time.

The exact solution to the problem of the heating of finite slabs and semi-infinite bodies by an arbitrary time-dependent heat flux is known, Refs. [48 to 50], and this solution can be expressed in terms of the solution for a constant heat flux. The special case of a parabolic heat input was treated by Kave and Yen, Ref. [51], while the general case of polynomial profiles was treated in Refs. [52, 53]. Sinusoidal and triangular heat pulses were investigated in Ref. [54]. These solutions were obtained by solving the usual classical heat conduction problem and are obtained in a form that is well suited for numerical evaluation. As a consequence,

an approximate method of solution is useful for obtaining the required temperature distributions in an expedient way.

The method to be employed here is Biot's variational principle for heat conduction. This principle and its application to problems with heat flux boundary conditions was discussed in detail in Ref. [13]. The present discussion of the principle applied to the problem of the heating of slabs with time dependent heat fluxes will be based upon Ref. [13]. It will be assumed that the reader is familiar with the method of solution outlined in Ref. [13].

The problem will be formulated for an arbitrary heat input and then particularized to the case of a triangular heat pulse. As was discussed in Ref. [13], the formulation of the problem for the heating of slabs proceeds in two phases.

#### Semi-Infinite Solid

The first phase of heating corresponds to the semi-infinite portion. This phase ends when the penetration depth is equal to the slab thickness. The temperature distribution is assumed to be parabolic, Fig. 1, and expressed in the form

$$0 = q_1 \left( 1 - \frac{x}{q_2} \right)^2$$

where  $q_1$  is the surface temperature and  $q_2$  is the penetration

depth. The generalized coordinate  $q_2$  is selected as the independent coordinate related to  $q_1$  through the expression for overall energy balance, Ref. [13]. Evaluation of the appropriate derivatives for the thermal potential and dissipation function, together with the thermal force and surface heat flux leads to the following expressions:

$$\partial V / \partial q_2 = c q_2^2 / 10$$

$$\partial D / \partial \dot{q}_2 = (c^2/k) q_1 q_2 [(1/42) \dot{q}_1 q_2 + (13/315) q_1 \dot{q}_2]$$

$$Q_2 = c q_1^2 / 10$$

$$H_n = (c/3) (\dot{q}_1 q_2 + q_1 \dot{q}_2)$$

Introduction of the flux condition

$$H_n = h q(t)$$

where  $q(t)$  is a dimensionless function of time, and the parameters

$$\eta = q_2^2 / b$$

$$\psi = q_1 k / h b$$

$$\tau = a t / b$$

will reduce the governing variational equation and flux condition to the following dimensionless equations:

$$15\dot{\psi}\eta^2 + 26\psi\eta\dot{\eta} = 147\psi \quad (22)$$

$$d(\eta\psi)/d\tau = 3q(\tau) \quad (23)$$

Equation (23) can be integrated immediately for the initial conditions  $\eta = \psi = 0$ , at  $\tau = 0$  and the function  $\psi$  eliminated between Eqs. (22) and (23). The resulting equation is

$$11\eta\dot{\eta} + 15\eta^2[q(\tau)/g(\tau) - g(0)] = 147 \quad (24)$$

where

$$g(\tau) = \int q(\tau) d\tau \quad (25)$$

and  $g(0)$  is the value of  $g$  at  $\tau = 0$ . If the substitutions

$$z = \eta^2$$

$$F(\tau) = q(\tau)/g(\tau) - g(0) \quad (26)$$

are made, Eq. (24) will reduce to the equation

$$\dot{z} + (30/11)zF(\tau) = (294/11) \quad (27)$$

The solution to Eq. (27) under the initial condition  $z = 0$  at  $\tau = 0$  is

$$z \equiv \eta^2 = b \int_0^\tau \exp[a \int F(\tau) d\tau] d\tau / \exp[a \int F(\tau) d\tau] \quad (28)$$

where  $a = (30/11)$  and  $b = (294/11)$ .

This solution gives the penetration depth as a function of time for arbitrary heating rates. Once  $\eta = \eta(\tau)$  is found, the surface temperature history can be found from Eq. (23). The result can be written in the form

$$\psi(\tau) = 3[g(\tau) - g(0)]/\eta \quad (29)$$

If the heating rate is such that  $g(0)$  equals zero, for example, a polynomial function, then Eq. (28) reduces to the form

$$\eta^2 = b \int_0^\tau [g(\tau)]^a d\tau / [g(\tau)]^a \quad (30)$$



In this case the solutions for the penetration depth and surface temperature, Eqs. (28) and (29), assume a simpler form. A particular form of interest for the heat input is the case where the heat input is of the form

$$q(\tau) = C\tau^n$$

that is, a heat flux dependent upon a power of the time. Evaluation of  $\eta^2$  from Eq. (30) yields

$$\eta^2 = 294\tau/41 + 30n \quad (31)$$

This result shows that the penetration depth is independent of the constant  $C$  in the heat input and varies as the square root of time. The factor of proportionality is dependent on the power of the time of the heat flux. Equation (31) is an interesting result that compares with the result obtained by Biot in Ref. [9].

In this reference, Biot found that if the surface temperature varied as  $\tau^n$ , the penetration depth relation is

$$\eta^2 = 147\tau/13 + 15n \quad (32)$$

These two results, Eqs. (31) and (32), show that the penetration depth varies as the square root of time for two cases; surface

temperature dependent on a power of time and heat flux dependent on a power of time.

If  $n = 0$  in Eq. (31), that is, a constant heat flux and  $n = (1/2)$  in Eq. (32), also a constant flux, the penetration depth relation is

$$\eta^2 = 294\tau/41$$

This result for constant heat flux was also noted in Ref. [13].

If  $n = 1$ , a linear heat input

$$q(\tau) = \tau/\tau_1 \quad (33)$$

where  $\tau_1$  equals a constant, the penetration depth relation is

$$\eta^2 = 294\tau/71 \quad (34)$$

The corresponding surface temperature history from Eq. (29) is

$$\psi(\tau) = (3/2\tau_1)(71/294)^{1/2} \tau^{3/2} \quad (35)$$

Equation (33) corresponds to the first leg of a triangular heat pulse and Eqs. (34) and (35) are the corresponding penetration depth and surface temperature histories. When  $\eta = 1$ , the

penetration depth equals the slab thickness  $b$ . The penetration time from Eq. (34) is

$$\tau_p = 71/294 \quad (36)$$

and the surface temperature at the penetration time is

$$\psi(\tau_p) = (3/2\tau_1)(\tau_p)^2 \quad (37)$$

Equation (37) gives the surface temperature at the penetration time which is to be used for the initial condition for phase 2, the finite slab.

#### Finite Slab

The temperature distribution for the second phase corresponding to the finite slab is shown in Fig. 1 and is written in the form

$$\theta = (a_1 m_2) \left(1 - \frac{x}{b}\right)^2 + a_3$$

If the analysis given in Ref. [13] for finite slabs is applied together with the parameters

$$\tau = at/b^2$$

$$\phi_{1,3} = q_{1,3} k/hb$$

the variational equation and flux condition become

$$10\dot{\phi}_1 + 32\dot{\phi}_3 = 84(\phi_1 - \phi_3)$$

$$\dot{\phi}_1 + 2\dot{\phi}_3 = 3q(\tau) \quad (38)$$

The flux equation, the second of Eqs. (38), can be integrated, and the function  $\phi_3$  eliminated between the equations to find

$$\dot{\phi}_1 + 21\phi_1 = 8q(\tau) + 21[g(\tau) - g(\tau_p)] + 7\phi_1(\tau_p) \quad (39)$$

Equation (39) can be integrated to yield

$$\phi_1 = \phi_1(\tau_p) + \int \exp(21\tau) \{8q(\tau) + 21[g(\tau) - g(\tau_p)] + 7\phi_1(\tau_p)\} d\tau / \exp(21\tau) \quad (40)$$

This result gives the surface temperature as a function of time for an arbitrary heat input. The rear temperature history can be found immediately from the flux condition. If the flux is a

polynomial, for example, Eq. (40) can be integrated in closed form for the surface temperature.

In summary then, the basic equations for an arbitrary heat input are given by Eqs. (28) and (29) for the semi-infinite solid and by Eq. (40) for the finite solid. These equations show that the solutions for an arbitrary heat input can be obtained in closed form for many cases. An example of the use of this method will be indicated to show how the surface temperature histories are found.

The example to be treated is the heating of a finite slab by a triangular heat pulse. This example was treated in Ref. [54].

#### Triangular Heat Pulse

The heat pulse to be considered is

$$q(\tau) = \begin{cases} \tau & 0 \leq \tau \leq 1 \\ 2 - \tau & 1 \leq \tau \leq 2 \end{cases} \quad (41)$$

The surface temperatures of a finite slab of thickness  $b$  are required for  $\tau \leq 2$ . The results for the semi-infinite portion are given by Eqs. (34) to (37) for  $\tau_1 = 1$ . The penetration time from Eq. (36) is

$$\tau_p = 71/204$$

This time corresponds to the starting time for the second phase.  
Equation (39) in this case reduces to

$$\phi_1 + 21\phi_1 = 10.5\tau^2 + 8\tau, \quad \tau > \tau_p$$

The solution is

$$\phi_1 = \frac{1}{2}\tau^2 - \frac{1}{3}\tau - \frac{1}{63} - 1.0056\exp(-21\tau) \quad (42)$$

from which the inner surface temperature is found

$$2\phi_3 = 1.5\tau^2 - \phi_1$$

or

$$\phi_3 = \frac{1}{2}\tau^2 - \frac{1}{6}\tau - \frac{1}{126} - 0.5028\exp(-21\tau) \quad (43)$$

Equations (42) and (43) are the surface temperature histories. These solutions apply until  $\tau = 1$  at which time the flux condition takes a different form from Eq. (41). At this time,

$$q(\tau) = 2 - \tau$$

and the governing equations, Eqs. (38), are again integrated for the initial conditions corresponding to  $\tau = 1$ . These conditions are evaluated from Eqs. (42) and (43) at  $\tau = 1$ . Evaluation leads to the values

$$\phi_1(1) = 0.81746$$

$$\phi_3(1) = 0.34127$$

to be used as the initial conditions. Solution of Eqs. (38) under these initial conditions, yields

$$\phi_1 = \frac{5}{3}\tau - \frac{1}{2}\tau^2 - \frac{20}{63} + 0.031746 \exp[-21(\tau-1)] \quad (44)$$

and

$$\phi_3 = \frac{13}{6}\tau - \frac{1}{2}\tau^2 - \frac{169}{126} + 0.015813 \exp[-21(\tau-1)] \quad (45)$$

Equations (44) and (45) give the surface temperatures of the slab for the time range  $1 \leq \tau \leq 21$ , while Eqs. (42) and (43) give the surface temperatures for the time range  $1 \leq \tau \leq \tau_p$ . The

inner surface temperature is of course equal to zero until the penetration time

The above results for the surface temperature are shown in Fig. 2 together with the results from Ref. [54]. It is seen from this figure that the present method is accurate as an approximate method of solution for the case of a triangular heat pulse



## SECTION VI. CONCLUDING REMARKS

A discussion and application of Biot's variational principle for heat conduction was presented in a previous paper, Ref. [13]. This reference contained a discussion of the principle from a mathematical viewpoint together with a discussion of the method of solution using the principle. A number of different one-dimensional heat conduction problems were treated to show the applicability of the method.

In the present paper, the literature discussing Biot's variational principle is reviewed, in addition to other variational principles and approximate methods of solution for heat conduction problems. The thermodynamic foundations of the variational principle are reviewed in Section IV showing that the variational principle for heat conduction has its basis in the physical concepts used in irreversible thermodynamics. The latter part of Section IV then shows how the form of the principle for heat conduction follows from the general formulation.

An application of the method employing the ideas given in Ref. [13] is presented in Section V. This application involves the heating of slabs exposed to time-dependent heat fluxes. For this type of heat flux, the present method of solution admits, in many cases, a closed form solution for the surface temperatures. The case of a triangular heat pulse is then presented in detail to show how the method is applied for problems of this type.

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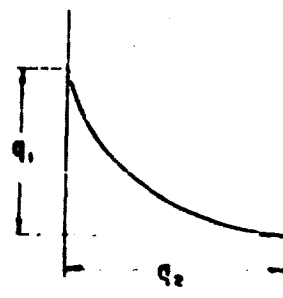
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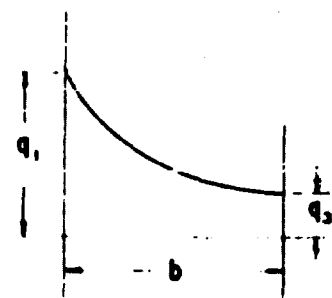
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SEMI-INFINITE BODY



FINITE SLAB

FIG. 1 TEMPERATURE PROFILES IN TERMS OF GENERALIZED COORDINATES; HEAT INPUT

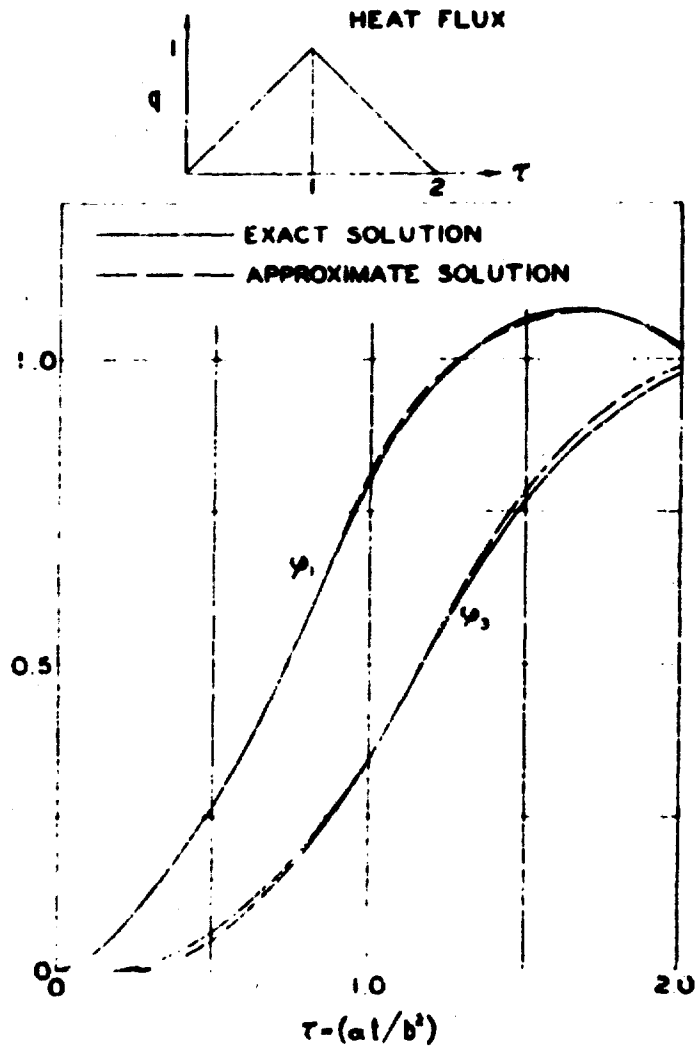


FIG 2 SURFACE TEMPERATURE HISTORIES

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